On the exactly-solvable pairing models for bosons.

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Abstract

We propose the new exactly solvable pairing model for bosons corresponding to the attractive pairing interaction. Using the electrostatic analogy, the solution of this model in thermodynamic limit is found. The transition from the superfluid phase with the Bose condensate and the Bogoliubov - type spectrum of excitations in the weak coupling regime to the incompressible phase with the gap in the excitation spectrum in the strong coupling regime is observed.

1. Introduction.

At present time the discrete-state BCS-type [1] pairing models attract much attention mainly in connection with the physics of ultra-small metallic grains (for a review see [2]). Previously, the discrete-state BCS model was solved by Richardson [3] in the context of nuclear physics. Later the integrability of the model was shown in ref.[4]. The BCS- type exactly solvable discrete pairing model for the system of bosons was first considered by Richardson [5]. In the continuum limit the condensate fraction and the Bogoliubov type spectrum of the low energy excitations (phonons) was obtained. Recently in ref.[6], [7] the pairing model for bosons in the context of finite system of bosons confined to a trap was considered. For this problem several generalizations of the simplest pairing model analogous to the BCS (fermionic) case (for example, see [8]) have been proposed. The remarkable feature of the pairing models for the confined bosons [6] is the phenomenon of Bose - condensation. In both cases at certain conditions another interesting phenomenon of the condensate fragmentation have been observed. From the theoretical point of view, the BCS-type pairing models are of interest due to their connection with the generalized Gaudin magnets, Knizhnik-Zamolodchikov equations, conformal field theory, and Quantum Inverse Scattering method (for example, see [9]) which was studied in a number of papers [10], [11]. From the mathematical point of view the models [5], [6] corresponds to the non-compact group $SU(1,1)$ which is the particular case of general non-compact $SU(n,m)$ groups [12]. The construction is equivalent to the non-compact $SL(2,R)$ spin chain with different infinite- dimensional representations at different sites.

In the present paper we propose and solve the simple modification of the model [5] corresponding to the attractive pairing interaction. Naively, for the attractive pairing interaction the ground state energy is a decreasing function of the particle number. However, one can consider the simple modification of the model [5] which has the correct behaviour of the ground

state energy both for the finite system and in the continuum limit. Namely, for the infinite - volume system of bosons we consider the Hamiltonian

$$H = \sum_p \epsilon_p n_p - g \sum_{p,p'} (a_p^+ a_{-p}^+)(a_{p'} a_{-p'}) + g' \sum_{p,p'} (a_p^+ a_p)(a_{p'}^+ a_{p'}), \tag{1}$$

where $g > 0$, $n_p = a_p^+ a_p$, a_p^+ (a_p) - are Bose creation (annihilation) operators corresponding to the plane waves with the momentum $p_a = 2\pi n_a/l$ ($a = x, y, z$, l- is the linear size of the system) and ϵ_p is the dispersion for the free particles (for example, $\epsilon_p = p^2/2m$). For pairing models for bosons confined in the external potential the indices in the Hamiltonian (1) should represent the states with definite principal quantum number and angular momenta [3]. For the system in the thermodynamic limit the sum over momenta p, p' in the second term of eq.(1) should be restricted to the values $|p|, |p'| < P$, where P is some cutoff $P \sim V$, and in order to have the correct behaviour of the ground state energy in the continuum limit, the constant g should be rescaled as $g \to g/V$, $V = l^3$. Although the last term in eq.(1) is nothing else but the constant equal to $g' N_b^2$, where N_b is the fixed number of bosons, the model have the correct ground state and in many aspects is a more realistic one in comparison with the model with repulsion considered by Richardson [5]. The model (1) can be applied both for the finite system of confined bosons and for the system in the thermodynamic limit. Note, that if one considers the model (1), as a result of truncation of the initial realistic interaction, in general, the terms of both type should be included. Note also that in the real systems like He^4 the attractive tail of the potential at large distances is always exist. For the finite systems, if the pairing interaction is considered as a residual interaction, the coupling constant can be of either sign. Previously, the modification of the model (1) for the case of attraction was studied numerically for the particular values of the parameters by Dukelsky and Schuck [6].

For the model (1) in the limit of the large number of bosons we find the excitation spectrum and the occupation probabilities for an arbitrary value of the coupling constant. As a function of the coupling constant g we observe the discontinuous transition between the two different regimes for the model (1). In the weak coupling regime there is the Bose condensate and the Bogoliubov-type spectrum of excitations. In the strong coupling limit the condensate is absent and there is a gap in the excitation spectrum. Qualitatively the results does not depend on the spacing and degeneracy of the energy levels and are valid in the limit of the large number of particles. The results are compared with the predictions of the mean- field theory approach and the Bogoliubov approximation. For the attractive model the naive mean - field approximation gives the exact results in thermodynamic limit in the case of the strong coupling, while the Bogoliubov approximation is exact in the weak coupling limit in some range of density depending on the coupling constant g. We show that the mean-field (variational) approach can be modified in order to take into account the Bose condensate and can be used for the model (1) to obtain the exact results in the thermodynamic limit in the whole range of parameters.

In Section 2 we review the exact solution of the model, and present the known generalizations of the model. We show that this class of models for bosons naturally appears in

the quasiclassical limit of the algebraic Bethe ansatz transfer matrix. We also present some new generalizations of the model [5] which can be useful for studying the superfluidity in the framework of this model. In Section 3 we present the solution of the model (1) in thermodynamic limit. In Section 4 we compare the exact solution with the predictions of the mean-field theory (variational) approach.

2. Pairing models for bosons.

The Hamiltonian for the boson pairing model has the form

$$H = \sum_{i=0}^{L-1} \epsilon_i n_i + g B^+ B^-, \tag{2}$$

where the coupling constant g is positive for the repulsion. The operators in (2) are defined through the pairs of bosonic creation and annihilation operators ϕ_{1i}^+, ϕ_{2i}^+ at the ith energy level ϵ_i ($\left[\phi_{\sigma i}; \phi_{\sigma i}^+\right] = 1$, $\sigma = 1, 2$). In terms of the pair creation and annihilation operators

$$b_i^+ = \phi_{1i}^+ \phi_{2i}^+, \qquad b_i = \phi_{1i}\phi_{2i},$$

the operators $B^\pm$ are defined as $B^+ = \sum_i b_i^+$, $B^- = \sum_i b_i$, and $n_i = n_{1i} + n_{2i}$ ($n_{\sigma i} = \phi_{\sigma i}^+ \phi_{\sigma i}$). For each energy level ϵ_i the operators $\phi_{\sigma i}^+$ describe the pair of states differing by the time reversal symmetry. For example, for the translationally invariant system the pair creation operator has the form $b_p^+ = \phi_p^+ \phi_{-p}^+$ (zero total momentum). An arbitrary degeneracy Ω_i of each energy level can also be taken into account. For the correct behaviour of the ground state one should re-scale $g \to g/L$, where L is the number of sites, in eq.(2). The rescaled value of g will be substituted in the final results throughout the paper. We define the particle density $\rho = N_b/L$, where N_b is the total number of bosons. The commutational relations for the pair creation and annihilation operators have the form

$$\left[(\phi_{1i}\phi_{2i}); (\phi_{1i}^+\phi_{2i}^+)\right] = 1 + n_i, \qquad n_i = \phi_{1i}^+\phi_{1i} + \phi_{2i}^+\phi_{2i}.$$

The commutational relations with the operator of the number of bosons n_i are:

$$\left[b_i^\pm; n_i\right] = \mp 2 b_i^\pm.$$

These commutational relations are equivalent to the group algebra of the pseudo-spin generators for the group $SU(1,1)$, which differs by the sign for the commutator $\left[S_i^+; S_i^-\right]$ from that of the $SU(2)$ algebra [12]. As in the case of the conventional discrete BCS-type models, the eigenstates can be constructed directly in terms of the operators

$$\Sigma^+(t) = \sum_i \frac{b_i^+}{(t - \epsilon_i)}, \qquad b_i^+ = \phi_{1i}^+ \phi_{2i}^+.$$

We seek for the eigenstates of the Hamiltonian (2) in the form:

$$|\phi(t)\rangle = \Sigma^+(t_1)\Sigma^+(t_2)\dots\Sigma^+(t_M)|\nu\rangle, \tag{3}$$

where the state $|\nu\rangle$ contains only the unpaired states i.e. defined by the conditions:

$$b_i|\nu\rangle = (\phi_{1i}\phi_{2i})|\nu\rangle = 0, \qquad n_i|\nu\rangle = \nu_i|\nu\rangle,$$

where ν_i are the (conserved) numbers of the unpaired bosons at the level i. Note that the state (3) is degenerate and does not determine the eigenstate completely. One should introduce the additional quantum numbers $\sigma_i = \pm 1$ such that for each site $(n_{1i} - n_{2i})|\nu\rangle = \sigma_i \nu_i |\nu\rangle$. The energy does not depend on σ_i, but the (angular) momentum depends on the quantum numbers σ_i. The complete set of states is given by the formula

$$\Big((b_0^+)^{n_0}(b_1^+)^{n_1}\dots(b_{L-1}^+)^{n_{L-1}}\Big)\,|(\nu_0,\sigma_0),(\nu_1,\sigma_1),\dots(\nu_{L-1},\sigma_{L-1})\rangle,$$

and can be characterized by integer quantum numbers n_i, $\sum_i n_i = M$, instead of the parameters t_i (apart from ν_i, σ_i). Since at $g \to 0$ the Hamiltonian reduces to the free one each eigenstate of (2) at finite g can be characterized by the integers $n_i^{(0)}$, $\sum_i n_i^{(0)} = M$, corresponding to the state at $g = 0$. In the limit $g \to 0$ the set of $n_i^{(0)}$ parameters $t_j \to \epsilon_i$. We use the following commutational relations for the operator $\Sigma^+(t)$ which can be proved using the commutational relations for b_i^+, b_i, n_i, and different from the formulas for the spin- 1/2 case:

$$n_i\Sigma^+(t) = \Sigma^+(t)n_i + \frac{2}{(t-\epsilon_i)}b_i^+, \qquad b_i\Sigma^+(t) = \Sigma^+(t)b_i + \frac{1}{(t-\epsilon_i)}(1+n_i). \tag{4}$$

For the Hamiltonian (2) the equations for the parameters t_i (3) are obtained in the same way as the formulas for the generalized Gaudin magnets (for example, see [5], [10]). The energy eigenvalues and the equations for the eigenstates are:

$$E = \sum_i \epsilon_i\nu_i + 2\sum_{i=1}^{M} t_i, \qquad \sum_\alpha \frac{\Omega_\alpha + \nu_\alpha}{t_i - \epsilon_\alpha} + \sum_{j\neq i}\frac{2}{t_i - t_j} = \frac{2}{g}. \tag{5}$$

The total number of bosons equals: $N_b = \sum_{i=0}^{L-1}\nu_i + 2M$. Note that for the Hamiltonian (2) the number of pairs and the number of the unpaired particles is conserved. Since the operator $\Delta n_i = n_{1i} - n_{2i}$ commutes with the generators of $SU(1,1)$ group one can add the term $\sum_i h_i \Delta n_i$ to the Hamiltonian (2) to obtain the model with the external field h_i. In this case the equations for t_i are the same as for the model (2), while the energy equals $E = \sum_i(\epsilon_i + h_i\sigma_i)\nu_i + 2\sum_i t_i$. The equations (5) are different from the equations for the BCS case by the normalization factors and the sign of the second term at the left-hand side. In the same way as in ref.[4], the set of commuting operators H_i $(i = 1,\dots L)$ and their eigenvalues can be found. In fact, analogously to the case of the $SU(2)$ group, consider the operators:

$$H_i = \frac{1}{g}n_i + \sum_{l\neq i}\frac{(S_iS_l)}{(\xi_i - \xi_l)} \tag{6}$$

where we have denoted by $(S_iS_j) = \sum_{a=x,y,z} S_i^a S_j^a$ and defined

$$S_i^+ = ib_i^+, \quad S_i^- = ib_i, \quad S^z = \frac{1}{2}(1+n_i). \tag{7}$$

Note that in terms of initial $SU(1,1)$ generators b_i^+, b_i, $1+n_{1i}+n_{2i}$, the scalar product has the form:

$$(S_iS_j) = -\frac{1}{2}(b_i^+b_j + b_j^+b_i) + \frac{1}{4}(1+n_i)(1+n_j).$$

Since the commutational relations for the operators S_i^a are the same as for the $SU(2)$ group, in analogy with the discrete - state BCS- model [4], [6], the operators (6) commute $[H_i; H_j] = 0$. At $\epsilon_i = \xi_i$ the linear combination $\sum_i \epsilon_i H_i$ gives the Hamiltonian (2) while for general $\epsilon_i \neq \xi_i$ we obtain the Hamiltonian depending on the two sets of parameters:

$$H = \sum_i \epsilon_i n_i + g \sum_{i<j} \frac{(\epsilon_i - \epsilon_j)}{(\xi_i - \xi_j)}(S_iS_j). \tag{8}$$

It was noted in ref.[6] that the choice $\xi_i = (\epsilon_i)^d$ leads to the realistic model for bosons confined in d - dimensional magnetic trap represented by the external harmonic well potential. The equations determining the eigenvalues of the Hamiltonian (8) and the operators (6) are given by the equations (5) with the parameters ϵ_i replaced by ξ_i.

Let us comment on the inclusion of the energy level which corresponds to the single Bose creation operator ϕ_0^+ ($p = 0$ level in the system with periodic boundary conditions or the $n = 0$ energy level in the spherically symmetric system) i.e. of the non-degenerate energy level. One can formally consider the states build up with *two* Bose creation operators of the form $(\phi_1^+\phi_2^+)^n|\nu_0\rangle$, where $\nu_0 = 0, 1$, and associate with this state the state $|\nu_0 + 2n\rangle$ of $\nu_0 + 2n$ bosons at the energy level 0. The interaction with the other pairs remains the same i.e. of the type $(\phi_1^+\phi_2^+)(\phi_{1i}\phi_{2i})$ $(i \neq 0)$. Thus the energy level $n = 0$ is considered on equal footing with the other energy levels with the exception of the allowed value of the parameter $\nu_0 = 0, 1$ which corresponds to the special type of interaction of pairs with the particles at the energy level 0.

Let us show that the discrete - state bosonic pairing models presented above as well as the new models with the interaction of pairs depending on the energy levels, can be obtained in the framework of the Quantum Inverse Scattering Method (for example, see [9]). Consider the Monodromy matrix defined in the usual way:

$$T_0(t) = K_0 L_{10} L_{20} \ldots L_{N0},$$

where $K_0 = \mathrm{diag}(e^{-\eta/2g}; e^{\eta/2g})$ is the usual diagonal twist matrix and the Lax operator obeying the Yang-Baxter equation is given by

$$L_{i0}(\xi_i, t) = \xi_i - t + \eta(\sigma S_i), \tag{9}$$

where the operators S_i^a $(a = x, y, z)$ were defined through the $SU(1,1)$ generators in the previous section, σ^a are the Pauli sigma matrices and ξ_i are the inhomogeneity parameters. Considering the quasiclassical limit $\eta \to 0$ of the transfer matrix $Z(t) = \mathrm{Tr}_0(T_0(t))$, we obtain the family of the Hamiltonians depending on the spectral parameter t, which commute at

different values of the parameters:

$$H(t) = \frac{1}{2g}\sum_i \frac{1}{(t-\xi_i)}(1+n_i) + 2\sum_{i<j}\frac{1}{(t-\xi_i)(t-\xi_j)}(S_iS_j), \tag{10}$$

$[H(t);H(t')] = 0$, where the expression for the scalar product (S_iS_j) was presented in the previous section. From eq.(10) taking the limits $t \to \xi_i$ or $t \to \infty$ the different pairing models for bosons can be obtained. The limit $t \to \xi_i$ corresponds to the operator H_i (6). The eigenvalues of $H(t)$ can be obtained from the eigenvalues of the transfer matrix $Z(t)$. We define the monodromy matrix in the following way:

$$T_0(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}_0,$$

and seek for an eigenstates in the form

$$|\phi(t)\rangle = B(t_1)B(t_2)\dots B(t_M)|\nu\rangle,$$

where the reference state $|\nu\rangle$ is defined by the conditions $S_i^-|\nu\rangle = 0$ and $n_i|\nu\rangle = \nu_i|\nu\rangle$. As in the usual $SU(2)$ case we observe that

$$C(t)|\nu\rangle = 0,$$

and the state $|\nu\rangle$ is an eigenvector of $A(t)$ and $D(t)$. The eigenvalues of the operators $A(t)$ and $D(t)$ are:

$$A(t)|\nu\rangle = \prod_\alpha \left(\xi_\alpha - t + \eta(1+\nu_\alpha)/2\right)|\nu\rangle, \quad D(t)|\nu\rangle = \prod_\alpha \left(\xi_\alpha - t - \eta(1+\nu_\alpha)/2\right)|\nu\rangle.$$

Following the well known procedure we obtain the Bethe ansatz equations:

$$e^{\eta/g}\prod_{\alpha=1}^{N}\left(\frac{t_i-\xi_\alpha+\eta(1+\nu_\alpha)/2}{t_i-\xi_\alpha-\eta(1+\nu_\alpha)/2}\right) = \prod_{\alpha\neq i}^{M}\left(\frac{t_i-t_\alpha-\eta}{t_i-t_\alpha+\eta}\right) \tag{11}$$

The corresponding eigenvalue of the transfer matrix $Z(t)$ equals

$$\Lambda(t) = \prod_i\left(\frac{t_i-t+\eta}{t_i-t}\right)\prod_\alpha(\xi_\alpha - t+\eta(1+\nu_\alpha)/2) + \prod_i\left(\frac{t-t_i+\eta}{t-t_i}\right)\prod_\alpha(\xi_\alpha - t-\eta(1+\nu_\alpha)/2),$$

where the two terms corresponds to the operators $A(t)$ and $D(t)$ respectively. Considering the quasiclassical limit of the Bethe ansatz equations, one reproduce the equations (5) for the eigenstates of the pairing Hamiltonian (2). The eigenvalues of the operators H_i and H (2) can be obtained from the last expression for $\Lambda(t)$. According to the general procedure [13] one can obtain the fundamental R - matrix corresponding to the direct product of two representations of the group $SU(1,1)$ (the so called fundamental Lax operator) which will lead to the new transfer matrix $Z^{(f)}(t)$ with the trace over the infinite- dimensional space, commuting with

the transfer matrix $Z(t)$ and the Hamiltonians of the new type, which is beyond the scope of the present paper. In order to obtain the trigonometric transfer matrix, one should have the special quantum group commutational relations $[S^+; S^-] = \sin(2\eta S^z)/\sin(\eta)$, which are not fulfilled for the $SU(1,1)$ generators. However, since the commutational relations for $S^{\pm}$, S^z coincide with the usual $SU(2)$ algebra, the Gaudin - type Hamiltonians [8], which are quadratic in the operators S^a, can be obtained in the trigonometric case. Thus all the known results, for the Gaudin- type Hamiltonians for the trigonometric and the elliptic cases, can be generalized to the case of $SU(1,1)$ - generators, constructed with the help of bosonic operators.

Let us briefly comment on the possible generalizations of these models. One can use the representation of spin-s operators through the single Bose creation and annihilation operators ϕ^+, ϕ, $[\phi; \phi^+] = 1$ as $S^+ = \phi^+(2s - \phi^+\phi)^{1/2}$, $S^- = (2s - \phi^+\phi)^{1/2}\phi$, $S^z = \phi^+\phi - s$, to include this spin in the quasiclassical Hamiltonian. In the fermionic case this leads to the generalized Dicke model if the limit $s \to \infty$ is taken. If one considers the limit $\xi_1 \to \infty$ for this single site in the hyperbolic version of the model (8) one can eliminate the terms, which do not conserve the number of bosons and obtain the model describing the interaction of single oscillator with the bosonic degrees of freedom:

$$H = \omega\phi^+\phi + \lambda(\phi^+\phi)\left(\sum_i \epsilon_i n_i + \sum_i c_i\sigma_i\nu_i\right) + \sum_i h_i\nu_i + \sum_i \epsilon_i n_i + gB^+B^-.$$

This Hamiltonian contains a number of free parameters which can be chosen in order to get the realistic model. In the sector with the oscillator excited to n-th energy level the model is reduced to the boson pairing model with the renormalized coupling constant g. At the same time the excitation energy (level spacing) for the oscillator depends on the average occupation numbers n_i for bosons. In contrast to the same model for fermions, the occupation probabilities $\langle n_i\rangle$ can be a small numbers, which allows for the small renormalization of the oscillator frequency. This model can be useful for studying the superfluidity without any assumptions.

3. Continuum limit.

Let us solve the model (2) in the continuum limit. Assuming that the distribution of roots t_i can be approximated by the continuous density $R(t)$, which is valid for the large number of pairs M, we get the following equation:

$$\int_a^b dt' \frac{R(t')}{t - t'} = f(t), \qquad f(t) = \frac{1}{g} - \frac{1}{2}\sum_\alpha \frac{C_\alpha}{t - \epsilon_\alpha}, \tag{12}$$

where the integral in a sense of principal value over the support of the function $R(t)$ is implied and $C_\alpha = \Omega_\alpha + \nu_\alpha$. According to [5] for the case of repulsion the ground state corresponds to the roots t_i located at the interval (ϵ_0, ϵ_1). One can argue that since at $g \to 0$ the ground state corresponds to all $t_i \to \epsilon_0$ and the roots t_i cannot cross the values ϵ_α for varying g, all $t_i \in (\epsilon_0, \epsilon_1)$. Thus one should solve the equation (12) assuming that the support of the function

$R(t)$ is the interval (ϵ_0, ϵ_1). The structure of the ground state for the repulsion and the general behaviour [5] of solutions of the equations (5) can be easily seen from the electrostatic analogy. Electrostatic analogy for the equations of the type (5) was previously used for the solution of the equations for the case of the BCS problem (for example, see [14]). One can consider the functional of the roots t_i as an energy of charges at the two-dimensional complex plane:

$$\Phi(t_i) = -\sum_{i,\alpha} C_\alpha \ln|t_i - \epsilon_\alpha| - 2\sum_{i<j} \ln|t_i - t_j| + (2/g)\sum_i \mathrm{Re}(t_i).$$

This energy functional corresponds to the repulsion of unit charges t_i and the repulsion of the charges t_i with the charges of the same sign $C_\alpha > 0$ placed at the fixed points ϵ_α. The external electric field of the magnitude $1/g$ is applied. The condition of stationary point (not minimum) of the functional $\Phi(t_i)$ with respect to the positions of the charges t_i leads to the equations (5). It is convenient to imagine the charges ϵ_α placed at the y axis of (x, y) plane as an energy levels. Then for the case of repulsion the external electric field is directed down, and each root gives the contribution to the energy equal to its height. One can see that all roots t_i are real, since due to the repulsion and the external electric field all the other configurations are unstable. Physically the picture is as follows. For each charge the repulsion due to the external charges ϵ_α below this charge, and the other roots below this charge, produce the force directed up. This force is compensated by the other charges above this charge and the external electric field directed down. For the ground state the roots t_i should be placed as low as possible. This picture allows one to use the physical intuition to find the solutions for the ground and the excited states of the model (2). For instance, the general behaviour of the solutions [5] is obvious.

Here we consider the pairing model (2) for the case of attraction $g < 0$ or equivalently the model (1) for $g > 0$. It was already mentioned that due to the additional term (1) the behaviour of the ground state energy as a function of particle number is correct. In many aspects the model (1) is more realistic in comparison with the model with repulsion [5]. For example, it has the Bogoliubov-type spectrum of excitations and the Bose condensate which varies continuously with the coupling constant from zero at some critical coupling g_c to N_b at $g = 0$. Later on we will omit the last term in eq.(1) in all the formulas. In the framework of electrostatic analogy the case of attraction corresponds to the external electric field directed up. Thus for any coupling constant g for the ground state all roots of the equations (5) located below the lowest energy level $\epsilon_0 = 0$. For small $|g|$ they are close to ϵ_0, while for large $|g|$ they are far below the level ϵ_0. At the sufficiently small $|g|$ the density of roots $R(t)$ grows at $t \to 0$ and bounded from below at some fixed point $-b$ ($b > 0$).

The simple method to find the solution for $R(t)$ (12) is, using the electrostatic analogy, to consider the electric field at the complex plane z produced by the unit charges t_i located at the interval $\Gamma = (a, b)$ of the real axis, the charges ϵ_α, and the external electric field:

$$h(z) = \int_a^b dt \frac{R(t)}{z - t} - f(z)$$

where the discontinuity $\Delta h(t)$ at Γ is given by the density of charges $R(t)$: $\Delta h(t) = h(t + i0) - h(t - i0) = 2\pi i R(t)$. The equation (12) takes the form $\bar{h}(t) = 0$, where $\bar{h}(t)$ is an average value of the field at both sides of Γ, and can be represented in the form:

$$\frac{1}{2\pi i} \oint_C dz \frac{h(z)}{z - t} = f(t) \tag{13}$$

for $t \in \Gamma$, where the contour C encloses the interval (a, b). For the sufficiently small coupling constant we use the following ansatz for the electric field $h(z)$ in the complex plane which has the branch cut along the interval (a, b) (in this case we take $a = 0$ and the interval $(-b, 0)$, $b > 0$ and use the coupling constant for the attraction $g > 0$):

$$h(z) = \sqrt{\frac{z + b}{z}} \left(\int_{-b}^{0} d\xi \frac{\phi(\xi)}{z - \xi} - \frac{1}{g} \right), \tag{14}$$

where the function $\phi(\xi)$ can be fixed from the condition for the residues of $h(z)$ at the points ϵ_α which are equal to $-C_\alpha/2$,

$$\phi(\xi) = \left(\frac{\xi}{\xi + b} \right)^{1/2} \rho(\xi), \qquad \rho(\xi) = -\frac{1}{2} \sum_\alpha C_\alpha \delta(\xi - \epsilon_\alpha).$$

The constant term in the parenthesis is fixed from the behaviour of the field at infinity, and the value of b is determined from the condition $\int dt R(t) = M$. The number of pairs and the energy $\Delta E = \sum_i 2t_i$ can be represented as an integrals in the complex plane over the contour C enclosing the interval Γ:

$$M = -\oint_C \frac{dz}{2\pi i} h(z), \qquad \Delta E = -\oint_C \frac{dz}{2\pi i} 2z h(z). \tag{15}$$

The integrals can be evaluated by means of deformation of the contour C into the small contours around the points ϵ_α and the large circle at the infinity. The equivalent way to find the energy is to substitute the ansatz for $h(z)$ into the equation (13), which after the deformation of the contour C allows one to find the function $\phi(\xi)$, presented above and the term $1/g$ in eq.(14). Using the same formulas for M and E (15), we obtain the following equation for the particle number:

$$\frac{b}{g} = N_b + L - \sum_\alpha C_\alpha S(\epsilon_\alpha), \qquad S(\xi) = \left(\frac{\xi}{\xi + b} \right)^{1/2}, \tag{16}$$

which determines the value of the parameter b. In contrast to the case of repulsion apriory we did not have any condition, which determines the upper bound for $|b|$: the support of $R(t)$ is not bounded from below for $g \to \infty$. The value of b found from the last equation should be substituted to the equation for the energy (15) which takes the form:

$$E = -\sum_\alpha \epsilon_\alpha + \sum_\alpha C_\alpha S(\epsilon_\alpha)(\epsilon_\alpha + b/2) - \frac{b^2}{4g}.$$

Using the equation (16) one can represent the last equation in a more convenient form:

$$E = -\sum_\alpha \epsilon_\alpha + \sum_\alpha C_\alpha E(\epsilon_\alpha) - \frac{b}{2}(N_b + L) - \frac{b^2}{4g}, \qquad E(\epsilon) = \sqrt{\epsilon(\epsilon + b)}. \tag{17}$$

In order to find the excitation spectrum and the occupation probabilities one should calculate the variation of the energy (17) over the quantum numbers ν_α and the energy levels ϵ_α respectively, taking into account the variation of the parameter b eq.(16). Let us note that the units for ϵ_i can be chosen in an arbitrary way, see eq.(5). The possible choice is $\epsilon_1 = 1/L$, such that $L\epsilon_1 = 1$. In thermodynamic limit there are two parameters - the density ρ and the coupling constant g. For example, one can imagine a one-dimensional lattice model with linear dispersion relation and L lattice sites. We will assume the units $L\epsilon_1 = 1$ and, as an example, consider the equal-spacing L level model with $\Omega_\alpha = 1$ and use the rescaled coupling constant $g \rightarrow g/L$ in the final results. We obtain from the equation (16) at $\nu_\alpha = 0$ the following equation for the parameter b:

$$b = g(f(b) + \rho), \tag{18}$$

where $f(b)$ is a smooth function which varies from zero to unity for $b \in (0, \infty)$. For example, for the equal-spacing model with $L\epsilon_1 = 1$ we have

$$f(b) = b\ln\left((1 + \sqrt{1+b})/\sqrt{b}\right) + 1 - \sqrt{1+b}.$$

Equation (18) gives the value of b as a function of the parameters g, ρ. First, consider the limit of the small coupling constant $g \ll 1$, $g\rho \ll 1$, such that $g \ll g\rho$. According to the last formula this limit corresponds to $b = g\rho$, and the high density limit $\rho \sim 1/g$. In this limit one can neglect the last sum in eq.(16) and obtain the excitation spectrum and the occupation numbers. Considering the energy (17) as a function of the quantum numbers ν_α and taking into account the variation of the parameter b according to eq.(16), we find the spectrum of phonons:

$$E(\epsilon) = \sqrt{\epsilon(\epsilon + g\rho)}.$$

As in ref.[5] one can show that the states corresponding to the excitation of pairs have the same energy, so that the (bosonic) quasiparticle interpretation of the excited states is true. This formula, corresponding to the particular limit $g \ll g\rho \ll 1$, is in agreement with predictions of the Bogoliubov approximation. However, in contrast to the repulsion, this spectrum is exact for an arbitrary value of the parameter $g\rho$ with respect to the energy level spacing ϵ_1, provided the condition $g \ll 1$ is satisfied.

Variation of the energy (31) with respect to the parameters ϵ_α gives the occupation probabilities $\langle n_\alpha \rangle$ which are different for $\langle n_0 \rangle$ (condensate) and $\langle n_i \rangle$, $i \neq 0$, which can be easily seen from the electrostatic analogy. In fact, if the parameter $g\rho$ is not too large, shifting the level $\epsilon_0 = 0$ down will obviously shift the distribution of roots and the lower boundary $-b$ down as

a whole, which means the existence of the condensate. Considering the variation $\delta E/\delta\epsilon_i$ for $i \neq 0$, we obtain:

$$\langle n_i \rangle = \frac{\epsilon_i + g\rho/2}{\sqrt{\epsilon_i(\epsilon_i + g\rho)}} - 1, \qquad i \neq 0. \tag{19}$$

The condensate fraction N_0 can be evaluated using the equation $N_0 = N_b - \sum_{i\neq 0}\langle n_i \rangle$. At $\epsilon_1 \ll g\rho$ the sum in (19) can be replaced by the integral, which gives:

$$N' = N_b - N_0 = L\left(\sqrt{1+g\rho} - 1\right),$$

The parameter which governs the condensate fraction is g: in the limit considered, $N' = (g\rho)L = gN_b \ll N_b$.

Next, consider the case $b \sim 1$. According to eq.(18) it is possible in the two cases: (i) $g \ll 1$ and $\rho \sim 1/g \gg 1$; (ii) $g \sim 1$, $\rho \sim 1$. In both cases calculating the excitation spectrum and the occupation numbers from the equations (16), (17), i.e. taking the variation of the energy (17) with respect to ν_α and ϵ_α (taking into account the variation of the parameter b according to eq.(16)) we obtain the expressions

$$E(\epsilon) = \sqrt{\epsilon(\epsilon + b)}, \qquad \langle n_i \rangle = \frac{\epsilon_i + b/2}{\sqrt{\epsilon_i(\epsilon_i + b)}} - 1, \qquad i \neq 0.$$

and the expression for the condensate

$$N' = N_b - N_0 = L(\sqrt{1+b} - 1). \tag{20}$$

Since $g\rho \sim 1$, in the case of large density $\rho \gg 1$ (case (i)) we will always have $N' \ll N_b$. However, in the case (ii), $\rho \sim 1$, for the coupling constant g larger than some critical value g_c, the last equation gives $N' > N_b$. That means that for the sufficiently large coupling constant the solution (14) is not correct.

Below we will show that at $g > g_c$ the solution should be modified. We will also show that the critical value g_c is determined exactly by the condition $N'(b) = N_b$, where b is the solution of the equation (18) (i.e. we will show that this condition coincides with the condition (26), see below). Here let us present the physical arguments, which show that at large g the new phase with the gap in the excitation spectrum should exist. As a limiting case, consider L (the large number) levels glued together. In this case the repulsion directed down is strong in comparison with the external field directed up and there cannot be roots t_i in the vicinity of $\epsilon_0 = 0$. Thus the support of $R(t)$ should be located far below $\epsilon_0 = 0$, at the distance of order $\sim gL$. The ground state energy will be of order $\sim -gLN_b/2$, and the gap in the excitation spectrum will exist. This picture is in agreement with the energy of the one-level model [5] with Ω replaced by L. So, as a first step, one could solve the one-level model with Ω replaced by L and $\nu = 0$ at $g \to \infty$, which would be the particular case of the general solution. In other words, at large g (weak external field) the distribution of charges will be unstable if the length $|b| \sim g\rho$ is much larger than the length $L\epsilon_1$.

Thus, we seek for solution of the equation (12) with the density support at the interval (b,a), $a,b<0$. In general one expects that since there is no external charges ϵ_α in the vicinity of the interval (b,a), the support of $R(t)$, it should be equal to zero at the endpoints. The numerical calculations suggest that for $|a|>0$ the function $R(t)$ is, in fact, equal to zero at the points a, b. It might seem that the ansatz for $h(z)$ should be chosen in such a way that as limiting case $a=0$ it would contain the solution for the interval $(b,0)$ i.e. in the form (14) with $a\neq 0$. However, we will show that correct solution reproduce eq.(14) at $a=0$. One can perform the calculations with the field of the type (14) and find that the parameters a, b are not completely fixed from the solution itself and one finds a number of functions $R(t)$ with different energy, which is not correct, as can be seen from the electrostatic analogy. Thus let us find the solution of (12) with the density $R(t)$ equal to zero at the endpoints of the interval (b,a), $a,b<0$. Since the field $h(z)$ should be a constant at the infinity, we consider the following function:

$$h(z)=\sqrt{(z-b)(z-a)}\left(\int_a^b d\xi\frac{\phi(\xi)}{z-\xi}\right), \tag{21}$$

where $\epsilon_0=0$ and $a,b<0$. The points a, b should be determined from the solution itself. Note that there are no poles other than the poles corresponding to the charges ϵ_i in $h(z)$. After changing the signs of the parameters a, b, from the equation (13) we find

$$\phi(\xi)=-\frac{1}{2}\sum_\alpha S(\xi)\delta(\xi-\epsilon_\alpha),\quad S^{-1}(\xi)=E(\xi)=\sqrt{(\xi+a)(\xi+b)}$$

and simultaneously the condition for the behaviour of the field at the infinity:

$$\int d\xi\phi(\xi)=-\frac{1}{2}\sum_{\alpha=0}^{L-1}S(\epsilon_\alpha)=-\frac{1}{g},$$

or, equivalently,

$$\sum_\alpha\frac{C_\alpha}{\sqrt{(\epsilon_\alpha+a)(\epsilon_\alpha+b)}}=\frac{2}{g}. \tag{22}$$

The first of the equations (15) gives

$$\frac{a+b}{g}=N_b+L-\sum_\alpha C_\alpha\epsilon_\alpha S(\epsilon_\alpha), \tag{23}$$

where the relation $M=(N_b-\nu)/2$ was used. Substituting the ansatz (21) to the second of the equations (15) and using the equation (22) we obtain the energy:

$$E=-\sum_\alpha\epsilon_\alpha+\sum_\alpha C_\alpha\epsilon_\alpha^2 S(\epsilon_\alpha)+\frac{1}{2}\sum_\alpha C_\alpha\epsilon_\alpha S(\epsilon_\alpha)(a+b)-\frac{(a-b)^2}{4g}.$$

Taking into account the equation (23) after some algebra this expression can be represented in the following form:

$$E=-\sum_\alpha\epsilon_\alpha+\sum_\alpha C_\alpha E(\epsilon_\alpha)-(N_b+L)\frac{a+b}{2}+\frac{1}{4g}(b-a)^2. \tag{24}$$

Thus the parameters a, b determined from the equations (22), (23) should be substituted to the energy (24). If the parameters a, b are fixed, if $a \neq 0$, the gap in the spectrum of excitations will appear and the Bose condensate will be absent. One can further rewrite the equations (23), (24) in order to see the similarity with the mean-field (variational) equations presented below. Introducing the notations

$$\mu = \frac{|a+b|}{2}, \qquad \Delta = \frac{|a-b|}{2},$$

the equations (23), (24) for μ, Δ at $\nu_\alpha = 0$ take the form:

$$N_b + L = \sum_i \frac{\epsilon_i + \mu}{\sqrt{(\epsilon_i+\mu)^2 - \Delta^2}}, \qquad \sum_i \frac{1}{\sqrt{(\epsilon_i+\mu)^2 - \Delta^2}} = \frac{2}{g},$$

$$E(\mu, \Delta) = -\sum_i \epsilon_i + \sum_i \sqrt{(\epsilon_i+\mu)^2 - \Delta^2} - (N_b + L)\mu + \frac{\Delta^2}{g}. \tag{25}$$

The gap in the excitation spectrum equals $\sqrt{\mu^2 - \Delta^2}$. The parameters μ, Δ found from the first two of the equations (25) should be substituted into the energy $E(\mu, \Delta)$ (25). The first two of the equations (25) are equivalent to the condition of minimum of the energy $E(a, b)$ over the variables a, b, which in terms of new variables reads $\delta E/\delta\mu = 0$, $\delta E/\delta\Delta = 0$. Thus the equations (25) are equivalent to the equations obtained from the mean-field theory (see below). The difference of the exact solution with the mean-field approach can appear only in the weak coupling regime in presence of the Bose condensate.

If the minimum of the energy exist, the solution of the equations (25) can be easily found. For example, for the equal-spacing L-level model with $L\epsilon_1 = 1$, taking the variations of (25) over μ and Δ we get the equations presented in the next section. The condition of the existence of the solution is

$$\sqrt{\mu^2 - \Delta^2} = \frac{1}{2(\rho+2)}\left(2C - \rho(\rho+2)\right) > 0, \qquad C = \frac{2+\rho}{e^{2/g} - 1}$$

Separately the parameters μ, Δ can be found from the equations

$$(\mu^2 - \Delta^2)^{1/2} = (C^2 - \Delta^2)/2C, \qquad \mu = (C^2 + \Delta^2)/2C.$$

For a given density ρ the last equation gives the critical value of the coupling constant g_c:

$$g_c(\rho) = \frac{2}{\ln(1 + 2/\rho)}. \tag{26}$$

For $g > g_c(\rho)$ the solution of the equations (25) exist, $|a| > 0$, and the gap in the energy spectrum $\sqrt{ab} = \sqrt{\mu^2 - \Delta^2} > 0$. The Bose condensate is absent in this phase. For $g = g_c(\rho)$ we have $a = 0$ and for $g < g_c(\rho)$ the gap closes and the solution (14) with the Bose condensate described above is valid. In fact, let us show that the critical value (26) coincides with the

value of g determined by the condition $N_b - N_0 < N_b$ in the framework of the solution (14) by the equation (18). One observes that eq.(16), (18) can be represented in the following form:

$$\frac{b}{g} = N_b - \sum_i \left(\frac{\epsilon_i + b/2}{\sqrt{\epsilon_i(\epsilon_i + b)}} - 1 \right) + \frac{b}{2} \sum_i \frac{1}{\sqrt{\epsilon_i(\epsilon_i + b)}}.$$

The cancellation of the first two terms at the right-hand side of this equation is equivalent to the condition $N' = N_b$ in the framework of the solution (21), while the last sum equals b/g in the framework of the solution (21) at $a = 0$. In fact, from eq.(25) at $a = 0$ ($\mu = \Delta$) we obtain exactly $2/g = \sum_i (\epsilon_i(\epsilon_i + b))^{-1/2}$. Therefore, two estimates of the critical point g_c found from two solutions in the weak and the strong coupling limits are coincide.

Let us show that at the value $a = 0$ the density $R(t)$ given by eq.(21), which was equal to zero at this point, is reduced to the density in the weak coupling regime (14) which is unbounded at $t = 0$. Due to the term $\sim 1/t$ in the parenthesis of eq.(21) one can rewrite the density (21),

$$R(t) = \frac{1}{\pi} \sqrt{(t - a)(t - b)} \left(-\frac{1}{2} \sum_\alpha \frac{S(\epsilon_\alpha)}{t - \epsilon_\alpha} \right),$$

in the following form:

$$R(t) = \frac{1}{\pi} \sqrt{\frac{t - a}{t - b}} \left(-\frac{1}{g} - \frac{1}{2} \sum_\alpha \frac{S(\epsilon_\alpha)(\epsilon_\alpha - a)}{t - \epsilon_\alpha} \right),$$

if the condition $\int d\xi \phi(\xi) = -1/g$ (22) is taken into account. The last expression coincides with the result obtained from the ansatz of the type (14) if the condition of minimum of the energy (22) as a function of a, b, $\sum_i (1/E(\epsilon_i)) = 2/g$ is satisfied. However, let us stress that the transition between the two phases at the critical point $g_c(\rho)$ is discontinuous.

Thus, we have shown that the transition from the strong coupling incompressible phase with the gap to the phase with the Bose condensate and the Bogoliubov- type spectrum of phonons takes place at the coupling constant $g = g_c(\rho)$ (26). At this point the condensate is equal to zero, $N_0 = 0$, but at $g < g_c(\rho)$ the condensate increases according to the equation (20) until the value $N_0 = N_b$ is reached at $g = 0$. Let us note that qualitatively these results are valid for the model with an arbitrary degeneracy of energy levels Ω_α and an arbitrary level spacing. Numerically the dependence $g_c(\rho)$ will have the different form. The limiting case of the solution at $|g| \to \infty$ coincides with the solution of the one-level model in this limit.

4. Mean-field solution.

Here we consider the mean field or variational solution of the pairing model (2) for the case of attraction:

$$H = \sum_{i=0}^{L-1} \epsilon_i n_i - g B^+ B^-, \qquad g > 0. \tag{27}$$

Let us describe the mean-field approach for the model (27) and find the range of the parameters for which the solution is exact in the thermodynamic limit. The mean-field Hamiltonian has the form

$$H_{MF} = \sum_i (\epsilon_i + \mu) n_i + \Delta \sum_i (b_i^+ + b_i) - \mu N_b + \frac{\Delta^2}{g}, \tag{28}$$

where μ is the chemical potential and the variational parameter Δ is real. The expression (28) can be considered in a sense of the Hubbard - Stratanovich transformation in the functional integral which can also be used to establish the validity of the mean-field theory. For the Hamiltonian (27) the mean field theory (28) is equivalent to the variational procedure with the trial wave function analogous to the BCS wave function. It is well known that the variational solution for the BCS Hamiltonian is exact in the thermodynamic limit (for example, see [15]). In contrast to the BCS case, for the bosonic model one has to introduce the chemical potential in order to fix the particle number. We show that in some range of parameters, at $g > g_c(\rho)$, the variational solution for the bosonic pairing model coincides with the exact solution presented above. At $g < g_c(\rho)$ the naive mean field solution is not correct. However, for our model one can modify the mean field (variational) approach taking into account the Bose condensation to obtain the exact results presented above in the whole range of parameters (except the extremely small coupling constant $g\rho \sim \epsilon_1$).

Each of the quadratic Hamiltonians H_i in the sum (28) can be diagonalized by means of the Bogoliubov transformation. For each site i introduce the new Bose creation and annihilation operators $\chi_{1,2}$, $\chi_{1,2}^+$ according to

$$\phi_1^+ = c\chi_1^+ + s\chi_2, \qquad \phi_2^+ = c\chi_2^+ + s\chi_1,$$

where the coefficients c_i, s_i are assumed to be real,

$$c_i^2 - s_i^2 = 1, \quad c_i = \mathrm{ch}(\phi_i), \quad s_i = \mathrm{sh}(\phi_i).$$

The expectation values of the particle number n_i and the energy H_i in the ground state are

$$\langle n_i \rangle = 2s_i^2, \qquad \langle H_i \rangle = (\epsilon_i + \mu)(c_i^2 + s_i^2 - 1) + \Delta 2 c_i s_i.$$

The condition of cancellation of the terms $\chi_1\chi_2$ and $\chi_1^+\chi_2^+$ takes the form:

$$\frac{2c_i s_i}{c_i^2 + s_i^2} = \mathrm{th}(2\phi_i) = -\frac{\Delta}{\epsilon_i + \mu}.$$

Thus we obtain the expressions for the energy and the number of particles as a functions of the parameter Δ and the chemical potential μ:

$$E_{MF}(\Delta) = \sum_i \left(\sqrt{(\epsilon_i + \mu)^2 - \Delta^2}(1 + n_i^\chi) - (\epsilon_i + \mu) \right) - \mu N_b + \frac{\Delta^2}{g},$$

$$N_b = \sum_i \left(\frac{|\epsilon_i + \mu|}{\sqrt{(\epsilon_i + \mu)^2 - \Delta^2}}(1 + n_i^\chi) - 1 \right), \tag{29}$$

where the operator n_i^χ equals

$$n_i^\chi = \chi_{1i}^+\chi_{1i} + \chi_{2i}^+\chi_{2i}, \qquad \nu_i\sigma_i = \chi_{1i}^+\chi_{1i} - \chi_{2i}^+\chi_{2i}.$$

The parameters Δ and μ should be determined from the condition of minimum of $E_{MF}(\Delta)$ (29) with the condition of fixed number of particles N_b. From eq.(29) the excitation energy $E_i = \sqrt{(\epsilon_i+\mu)^2 - \Delta^2}$. The ground state corresponds to the quantum numbers $n_i^\chi = 0$, or, equivalently, to the state $|0\rangle_\chi$ annihilated by the operators $\chi_{1,2}$:

$$\chi_{1i}|0\rangle_\chi = 0, \quad \chi_{2i}|0\rangle_\chi = 0, \quad i = 0, \dots L-1.$$

In terms of the initial operators ϕ_1^+, ϕ_2^+ this state can be represented as

$$|0\rangle_\chi = \prod_i e^{\alpha_i(\phi_{1i}^+\phi_{2i}^+)}|0\rangle, \quad \alpha_i = \frac{s_i}{c_i} = \mathrm{th}(\phi_i), \tag{30}$$

where $|0\rangle$ is the vacuum with respect to the initial operators: $\phi_{1i}|0\rangle = 0$, $\phi_{2i}|0\rangle = 0$. In fact, for each i for the state $|\alpha\rangle = \exp(\alpha\phi_1^+\phi_2^+)|0\rangle$ one finds

$$(\phi_1 - \alpha\phi_2^+)|\alpha\rangle = 0, \qquad |\alpha\rangle = e^{\alpha(\phi_1^+\phi_2^+)}|0\rangle.$$

Substituting the expressions for the operators ϕ_1, ϕ_2, one can see that the state (30) is annihilated by the operators χ_1, χ_2 provided the condition $\alpha = s/c = \mathrm{th}(\phi)$ is satisfied. The excited states can be constructed from the state (30) by action of the operators χ_1^+, χ_2^+:

$$(\chi_1^+)^{n_1}(\chi_2^+)^{n_2}|0\rangle_\chi, \qquad \nu = |n_1 - n_2|.$$

Let us show that for the model (27) the mean-field theory approach is equivalent to the variational procedure with the trial variational wave function of the form (30). Although this wave function does not correspond to a definite particle number, it can be fixed in an average as in the usual BCS theory, which is justified in the continuum limit. Expectation value of the Hamiltonian (27) over the state (30) as a function of the variational parameters ϕ_i takes the form:

$$E = \sum_i (\epsilon_i + \mu)(2s_i^2) + g\left(\sum_i c_i s_i\right)^2 - \mu N_b. \tag{31}$$

Taking the variation of (31) with respect to ϕ_i one finds the equations presented above with

$$\Delta = g\sum_i (c_i s_i).$$

The average particle number $N_b = \sum_i 2s_i^2$. The existence of the condensate means $\phi_0 \to \infty$. Substituting this value to the right -hand side of eq.(31) and assuming $N_0 = N_b$, one finds $\mu = g\rho/2$, and the spectrum $E(\epsilon) = \sqrt{\epsilon(\epsilon + g\rho)}$ in agreement with the Bogoliubov approximation. Thus although the Bogoliubov approximation corresponds to the variational estimate of the energy, in general, it does not correspond to the minimum of the energy on the class of the

wave functions (30). However we show that in the weak coupling limit the naive variational approach does not lead to the correct results while the Bogoliubov approximation gives the exact results in the weak coupling limit if the density is not too small ($1 \ll \rho \ll 1/g$).

In the strong coupling regime $g > g_c(\rho)$ the equations (29) coincide with the exact equations obtained in section 6. The equation $\delta E_{MF}(\Delta)/\delta\Delta = 0$ together with the second of the equations (29) allows one to find the parameters μ, Δ, the occupation numbers and the gap in the energy spectrum. In particular, for the equal-spacing L level model with $L\epsilon_1 = 1$ the equations take the form:

$$\ln\left(1 + \mu + \sqrt{(1+\mu)^2 - \Delta^2}\right) - \ln\left(\mu + \sqrt{\mu^2 - \Delta^2}\right) = \frac{2}{g}, \tag{32}$$

$$\rho + 1 = \sqrt{(\mu+1)^2 - \Delta^2} - \sqrt{\mu^2 - \Delta^2}, \tag{33}$$

which have the solution found in section 6. The results are in agreement with the exact solution.

Let us see if the exact solution in the weak coupling case $g < g_c$ can be obtained in the framework of the mean-field (variational) approach. To get the expectation value $\langle n_0 \rangle = N_0$ of order N_b, one should take $\Delta = \mu + \delta$, where δ is the small parameter of order $1/L$. Substituting the value $\mu = \Delta$ into one of the equations (32), (33) one finds that the result for Δ contradicts the exact solution. Thus the naive mean field approach fails for the region of the parameters where the solution of the equations $\delta E_{MF}/\delta\mu = 0$, $\delta E_{MF}/\delta\Delta = 0$ does not exist. To get the correct results, one should use the following method. First, substitute the parameter $\mu = \Delta$ into $E_{MF}(\mu, \Delta) \to E_{MF}(\Delta, \Delta)$ (29). Then the solution of the equation $\delta E_{MF}/\delta\Delta = 0$ gives the results in agreement with exact solution of section 6. In fact, one can see that the variation of this function leads to the equation (16), which in the framework of the exact solution was used to determine the parameter b. The validity of this method can be shown in the same way as for the usual BCS model. In the framework of the functional integral approach the factor L (the volume) appears in the exponent in front of the action if the condensate is absent. To take into account the condensate one can introduce the δ- function of the form

$$\delta(n_0(\mu, \Delta) - N_b + N'(\Delta)),$$

where the function $n_0(\mu, \Delta) = \langle n_0 \rangle$ is given by eq.(29) for $i = 0$ and the function $N'(\Delta)$ is determined by the sum $\sum_{i \neq 0} \langle n_i \rangle$, with $\langle n_i \rangle$ given by eq.(29) with $\mu = \Delta$. This factor will give $\mu = \Delta$ with the accuracy of order $1/N_b$ and remove the integration over μ. If the saddle point for the remaining integration over Δ exist and gives the value $N'(\Delta) < N_b$, which indeed takes place, the solution is exact in the thermodynamic limit. The particle number is correctly fixed within this approach. The same can also be shown using the trial variational wave function of the form $|N_0, \phi_1, \ldots \phi_{L-1}\rangle$, where N_0 and ϕ_i, $i \neq 0$ are the variational parameters. Thus the modified mean-field approach is valid in the whole range of the parameters with the exception of the extremely small coupling constant $g\rho \sim \epsilon_1$, when the value of N', the number of particles out of the condensate, becomes of order of unity.

Conclusion.

In the present paper we have shown that the discrete-state BCS-type pairing models for bosons can be considered as a quasiclassical limit of the eigenvalue problem of the general transfer matrix in the framework of the algebraic Bethe ansatz method. We introduced the new pairing model for bosons corresponding to the attractive pairing interaction. It was shown that the weak coupling phase, $g < g_c$, is characterized by the Bose condensation and the Bogoliubov-type spectrum of phonons. In the strong coupling phase at $g > g_c$ the Bose condensate is absent and there is a gap in the excitation spectrum. Note that the transition of this type from the incompressible Mott insulating phase to the superfluid phase is usually expected in the Bose Hubbard model. We have shown that naive variational approach is not applicable in the weak coupling limit at $g < g_c$, when the condensate fraction exist. However, for our model one can modify the variational procedure taking into account the condensate fraction to obtain the exact solution in the whole range of parameters. The Bogoliubov approximation gives the correct results in agreement with the exact solution in the limit $g \ll 1$ and $1 \ll \rho \ll 1/g$, such that the parameter $g\rho \ll 1$, i.e. when the parameter $b = g\rho$ (see eq.(18)). The proposed model with an attractive pairing interaction can be interesting both in the context of applications to the finite systems of the confined bosons and for studying the phenomenon of superfluidity in the exactly- solvable model.

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